Development of Numerosity Estimation: A Linear to Logarithmic Shift?

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Abstract

Young children's estimates of numerosity increase approximately logarithmically with actual set size. The conventional interpretation of this finding is that children's estimates reflect an innate logarithmic encoding of number. A recent set of findings, however, suggest logarithmic number-line estimates could emerge via a dynamic encoding mechanism that is sensitive to the prior distribution of stimuli. Here we test this idea by examining trial-to-trial changes in logarithmicity of numerosity estimates. Against the dynamic encoding hypothesis, first trial estimates in both adults (Study 1) and adults and children (Study 2) were strongly logarithmic, despite there being zero previous stimuli. Additionally, although numerosity of a previous trial affected adult estimates of numerosity, the nature of this effect varied across experiments, yet always resulted in a logarithmic-to-linear shift from trial-to-trial. These results suggest that a dynamic encoding mechanism is neither necessary nor sufficient to elicit logarithmic estimates of numerosity.

Keywords: cognitive development; numerical cognition; spatial cognition; numerosity perception

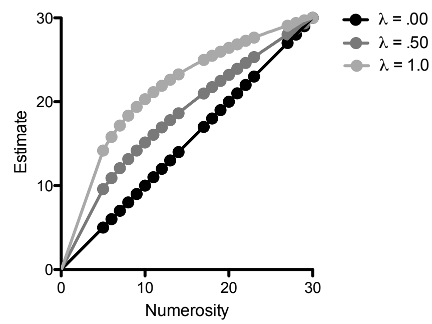

Figure 1: In the "logarithmic-to-linear shift", children's estimates (shown in greys) initially have a strong logarithmic component (λ = 1), whereas older children's and adults' estimates (shown in black) are approximately linear (λ = 0).

Introduction

Mapping numbers to space is fundamental to measurement and mathematics. While spatial-numeric associations are evident in infant humans and other animals (see McCrink & Opfer, 2014, for review), the nature of this mapping changes dramatically with age and education (Ashcraft & Moore, 2012; Booth & Siegler, 2006; Dehaene, Izard, Spelke, & Pica, 2008; Geary, Hoard, Nugent, & Byrd-Craven, 2008; Gunderson, Ramirez, Beilock, & Levine, 2012; Hurst, Leigh Monahan, Heller, & Cordes, 2014; Siegler & Opfer, 2003).

Developmental changes in number-to-space mapping is perhaps most evident in number-line estimation, in which subjects estimate the location of a number (an Arabic numeral or a number of dots) on a line flanked by two other numbers (e.g., 0 and 30). In younger children, the psychophysical function relating numeric value to spatial estimate has been found to have a larger logarithmic component (λ) than in older children and educated adults (the "logarithmic to linear shift," Siegler, Thompson, & Opfer, 2009, shown in Fig. 1) . Thus, among young children (Booth & Siegler, 2006; Siegler & Opfer, 2003) and Amazonian indigene (Dehaene, Izard, Spelke, & Pica, 2008), for example, the number-to-space mapping is strongly logarithmic (e.g., placing 15 past the midpoint of a 0-30 number line); however, with age and schooling, the mapping becomes approximately linear. Linearity of number-line estimation is important because it predicts several numeric outcomes, including number memory (Thompson & Siegler, 2010), number categorization (Opfer & Thompson, 2008), dyscalculia (Geary, Hoard, Nugent, & Byrd-Craven, 2008), math scores (Fazio, Bailey, Thompson, & Siegler, 2014; Gunderson, Ramirez, Beilock, & Levine, 2012), and quality of economic utility judgments (Schley & Peters, 2014).

Why might early, untutored number-line estimates be logarithmically compressed? One idea is that the logarithmic pattern of number-line estimates for symbolic numbers reflects the subjective similarity of the numeric magnitudes given by the perception of non-symbolic numbers (Dehaene, 2003, 2007; Nieder & Merten, 2007; Siegler & Opfer, 2003). That is, just as children (and other animals) perceive 14 dots as being more similar to 19 dots than to 9 dots (Fechner's Law), so they place 14 closer to 19 than to 9 on a number line. This logarithmic pattern of number-line estimates has also been observed in educated adults under certain circumstances. For example, under attentional load, adults' estimates of numerosity also increase in logarithmicity (Anobile, Cicchini, & Burr, 2012). Further, as adults make number-line estimates, their reach toward the number-line initially points to a logarithmic position before being corrected toward a linear position (Dotan & Dehaene, 2013). Together, these results point to the logarithmic number-line mapping as a reflection of the native representation of numerosity, one that can be extended to numeric symbols–as well as supervened–with a combination of education and attentional effort.

Might Dynamic Encoding Yield a Linear-to-Logarithmic Shift?

A recent proposal has challenged this idea of an untutored compressive mapping of number to space. According to this proposal, compressive mapping reflects only a dynamic encoding mechanism, such that the estimated magnitude of a number is influenced by the number previously encountered, which tends to anchor the next estimate (Cicchini, Anobile, & Burr, 2014). That is, a logarithmic pattern of estimates on a number line does not come from a "static logarithmic transform," but emerges online from a "central tendency of judgment," such that responses tend to be biased toward the mean of a stimulus distribution. Thus, over the course of many number-line estimates, what emerges is a linear-to-logarithmic shift, rather than a logarithmic-to-linear shift. If correct, this proposal is important in that it calls for a fundamental reinterpretation of more than a decade of research.

To test this dynamic encoding hypothesis, Cicchini et al. (2014) asked five adults to estimate the position of a numerosity (a set of dots) on a number line, with 9 unique numerosities tested on a total of 144 trials (Fig. 2) . Consistent with a central tendency of judgment effect, adults tended to underestimate the position of a number if the previous number was small and to overestimate the position if the previous number was large. This serial dependency was strongest in a dual-task condition in which adults were asked to perform a number-line task along with a color conjunction task. Additionally, this dual task condition yielded estimates with a higher logarithmic component (λ = .38) than the single task condition (λ =.11). These findings led the authors to conclude, "the strongest evidence for logarithmic coding [of number] was the logarithmic number line: Because that now has a more plausible explanation, there exists no evidence at all for logarithmic encoding of number in primate brains" (p. 7871).

The Present Studies

In this paper we revisit whether any causal relation exists between dynamic encoding and compressive numerosity estimates. Although Cicchini et al. (2014) have shown that attentional load increases both logarithmicity of number-line estimates and responses to serial dependencies, it remains unclear whether dynamic encoding is either necessary or sufficient for logarithmic number-line estimates.

To test whether dynamic encoding is *necessary* for logarithmic number-line estimates, we were particularly interested in examining subjects' estimates on the first trial of the number-line task. On the first trial, no dynamic response is possible because no previous number had been encountered. Thus, the predictions of the two accounts diverge most strongly for first trial responses. If numeric magnitudes were initially encoded logarithmically, a strong logarithmic component should be evident on the first trial. On the other hand, if compressive mappings simply reflect a central tendency to judgment, no logarithmic component should be evident.

To examine whether dynamic encoding is *sufficient* for logarithmic number-line estimates, we also examined the logarithmic component of number-line estimates as a function of (1) trial position, which necessarily changes the distribution of previous stimuli, and (2) the overall number of trials, such that any cumulative dynamic effect could be increased. If dynamic encoding were sufficient for compressive mappings, a linear-to-logarithmic shift would be expected, with the logarithmic component of number-line estimates increasing steadily after the first trial. In contrast, if experience on the task simply improves familiarity with the numbers or increases attention to the task, a standard logarithmic-to-linear shift would be expected as trial position increased.

We explored these issues in adults alone (Study 1), as well as children and adults (Study 2). An important difference between the two studies concerned the number of trials. Study 1 was a replication attempt of Cicchini et al. (2014), in which the same numerosity was presented repeatedly. Study 2 followed the more conventional design of number-line tasks, which present each stimulus only once (Siegler & Booth, 2004; Siegler & Opfer, 2003).

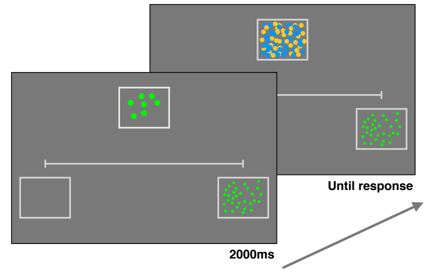

Figure 2: Illustration of a number-to-space mapping task

Experiment 1

In Experiment 1, we sought to replicate the central tendency of judgment effect with a larger number of adult participants (n = 20) than previously tested (Cicchini et al., 2014; n = 5). This increase in sample size was necessary to obtain the statistical power necessary to make meaningful inferences about single trial judgments.

Methods

Subjects Twenty undergraduate students at the Ohio State University participated in the study (*M* = 19.59 years, *SD* = 1.29 years).

Materials And Procedure Participants were given a non-symbolic number-line task in which they were shown a set of dots (5 - 29) on a computer screen and asked to estimate the number of dots by mouse-clicking a position on a number line (a line flanked by 0 dots on the left and 30 dots on the right) (Fig. 2) . On each trial, the set of dots (5 - 29) to be estimated was shown briefly (2000 ms) and immediately followed by a random-noise mask; this procedure was employed to prevent counting. This procedure continued for 3 blocks of 20 trials. The to-be-estimated numerosities were chosen to sample the

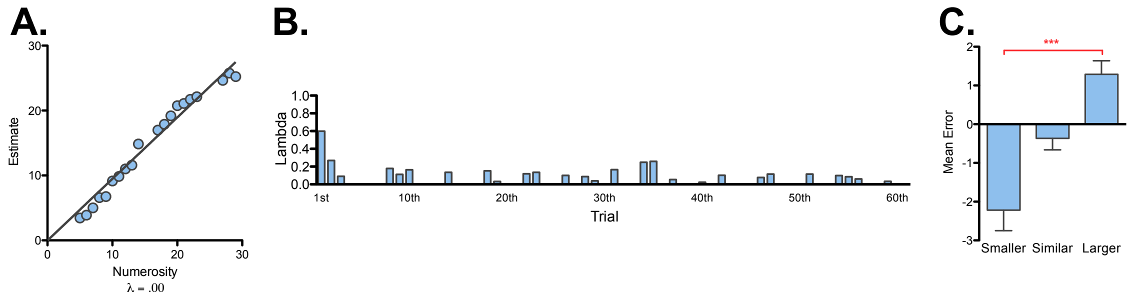

Figure 3: Experiment 1, A: Median estimates collapsing over all trials. B: Logarithmic component (λ) over all trials. C: Effect of magnitude of the previous trial.

non-subitizable numbers evenly: 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 17, 18, 19, 29, 21, 22, 23, 27, 28, 29. Compared to the previous study, in which only 9 numerosities from 0 to 1000 were used (Cicchini et al., 2014), the current study included most of the numerosities between 0 and 30 except for those that were subitizable or might serve as landmarks. By doing so, we examined mappings of all possible numerical values onto a number line and their trial-by-trial changes more thoroughly. The order of numerosities presented was determined by a Latin square, such that each numerosity was presented to one subject on each of the 20 trials. After instructions, participants started the task with no practice trials or feedback of any kind.

Results and Discussion

We first examined whether dynamic encoding was necessary to elicit logarithmic numerosity estimates. To address this, the positions of participant's estimates were regressed against the number of dots presented. For this purpose, we used Cicchini et al.'s (2014) combined log-linear regression model:

$$R = a((1-\lambda)N + \lambda \frac{N_{max}}{\ln(N_{max})} \ln(N)),$$

where R denotes the response to given numerosity N, a is a scaling factor, and N_{max} is the numerosity at the right end of a number line (30 in the current study). The degree of logarithmicity is denoted by λ (Anobile, Cicchini, & Burr, 2012; Cicchini, Anobile, & Burr, 2014). If λ equals 0, the estimates are perfectly linear, whereas if λ is 1, the estimates are perfectly logarithmic (Fig. 1) Collapsing over all trials, the median estimates were completely linear (λ = .00) (Fig. 3A) .

Weights of the logarithmic component, λ, were tracked on a trial by trial basis. Against the idea that dynamic encoding is necessary for logairthmic estimates, a large logarithmic component was evident on the first trial, but decreased steadily to the last trial (λ = .60 for the first, λ = .00 for the last trial) (Table 1 and Fig. 3B). To test this observation statistically, we regressed trial number against the logarithmicity index. Consistent with the nominal values, logarithmicity index values reliably declined with trial number (b = -.003, $p < .05$). These results indicate that logarithmic number-line estimates do not require any dynamic response to previous stimulus distributions; rather, they are strongly present on the first trial. Additionally, if subjects' estimates were subject to any serial dependencies, the effect of these were to elicit the standard logarithmic-to-linear shift, rather than to elicit the linear-to-logarithmic shift envisioned by the dynamic encoding hypothesis.

We next examined whether any central tendency of judgment effect was even present. To test this, we first followed Cicchini et al.'s (2014) procedure in which errors for each trial were categorized into 3 groups based on magnitude of the previous stimulus (larger by 5, similar, or smaller by 5). Consistent with a dynamic encoding effect (Fig. 3C) , we found that when a set of dots was presented after a larger set, subjects tended to overestimate the number of dots. Also, when a set of dots was presented after a smaller set, subjects tended to underestimate the number of dots. Thus, like Cicchini et al. (2014), we found that the average error of the estimate differed reliably as a function of magnitude of the numerosity prior to the current one ($F(2, 20) = 24.34$, $p < .001$). Thus, although a quite strong central tendency of judgment was evident in our study, this tendency seemed to have the effect of increasing the linearity of estimates.

Table 1: Logarithmic component λ in Exp. 1 and Exp. 2.

	Group	First Trial	Last Trial	All Trials
Exp.1				
	Adult	.60	.00	.00
Exp.2				
	Child	1.00	1.00	1.00
	Adult	.70	.18	.11

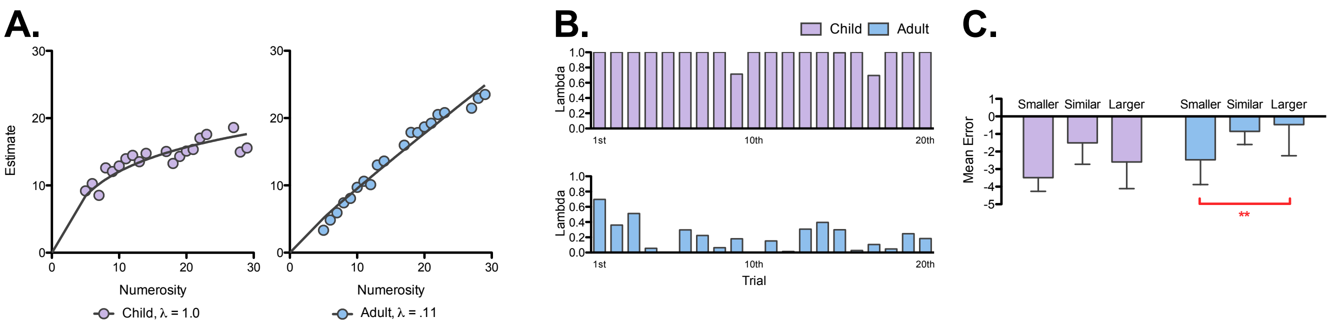

Figure 4: Experiment 2, A: Child and adult median estimates collapsing over all trials. B: Logarithmic component (λ) over all trials. C: Effect of magnitude of the previous trial in child and adult number line estimation.

Experiment 2

Experiment 2 examined whether dynamic encoding might also affect children's estimates of numerosity on the number line. In previous studies of numerosity estimation using the number-line task, estimates tended to become more linear with age. The results of Study 1 suggest that this age difference might be the result of dynamic encoding mechanisms increasing the linearity of adults' otherwise logarithmic estimates. An important difference from Study 1, however, is that the mapping task was shortened to a block of 20 trials, which is more typical of number-line studies. If log-like compression found in the previous work with less than 20 trials were attributed to dynamic effects, the central tendency of judgment effect would be expected to appear in the shorter task as well.

Methods

Subjects Seventy 5- to 6-year-old children in Columbus (M = 6.02 years, SD = .45 years) and 80 undergraduate students at Ohio State (M = 19.75 years, SD = 2.37 years) were recruited for the experiment.

Materials And Procedure Participants were asked to complete a number line estimation task consisting of one block of 20 trials. The stimuli were identical to those in Study 1, but each stimulus was presented only once. Adult participants were instructed to position where the number went on a number line by clicking a mouse. Children were told to point to the place where the number belonged, and a female experimenter assisted with a mouse click.

Results and Discussion

Logarithmic components were computed as in Study 1. As illustrated in Table 1 and Figure 4A-B , children's estimates of numerosity were completely logarithmic (λ = 1) on the first trial, the last trial, and nearly every trial in between. In contrast, adults' estimates were somewhat linear when collapsing over all trials (λ = .11), where our observed lambda value was identical to that observed by Cicchini et al. (2014). These results narrowly replicate previous findings demonstrating age differences in numerosity estimation (Siegler & Opfer, 2003). However, on the first trial, the logarithmicity of adults' and children's estimates (.70 for adults vs. 1.00 for children) were much closer in value than was evident on the last trial (.18 for adults vs. 1.00 for children). This difference reflects the fact that there was a trend for the logarithmicity of adults' estimates to decrease from trial to trial (b = -.01, p = .10).

Overall, these results were not consistent with the dynamic encoding explanation for compressed number-to-space mapping: compression was again evident on trial 1, before any dynamic encoding mechanism could have an effect. Nor were these results consistent with the idea that the representation of numerosity changes much from childhood to adulthood: on the first trial, adults' estimates were about as logarithmic as childrens' estimates. Rather, trial-to-trial analyses suggest that the native representation of numerosity is logarithmic, but this impression can be supervened (at least in adults) by repeatedly encountering numerosities.

Did these repeated encounters with numerosities result in a central tendency of judgment, as predicted by the dynamic encoding hypothesis? To test this, response errors were again grouped by the magnitude of the previous trial (Fig. 4C) . The central tendency could not be found in children's responses, though adult mappings changed significantly as a function of the previous trial ($F(2, 20) = 8.52$, $p < .01$). Specifically, a number position was more underestimated when the previous number was smaller than when it was similar or larger. Unlike Cicchini et al. (2014) or Study 1, however, overestimation after the larger previous number was not observed. To examine this more closely, estimates were regressed on magnitude of the previous trial to examine effects of trials given in the past and future. From a significance test by bootstrapping, we again found a significant influence of the immediately previous trial (β = .08, $p < .01$ for -1 trials). However, magnitude of the trials presented 6, 8, and 18 trials ago also significantly affected the position of a current numerosity, but in an opposite way (β = -.08, $p < .001$ for -6 trials, β = -.10, $p < .001$ for -8 trials, and β = -.21, $p < .05$ for -18 trials): overestimation after the smaller previous numbers and underestimation after the larger previous numbers. This significant but negative association of a current response with previous trials presented as much as 18 trials ago suggests that dynamic effects may be statistical artifacts at least in a short and traditional version of the number line task.

General Discussion

Whether estimating the number of dots (Booth & Siegler, 2006) or the value of an Arabic numeral (Siegler & Opfer, 2003), the estimates of young children and unschooled Amazonian indigine tend to increase logarithmically with actual value, whereas older children's and adults' estimates tend to increase linearly with actual value ("the logarithmic to linear shift"; Siegler, Thompson, & Opfer, 2009). This logarithmic mapping of number to space is not unique to young children and unschooled adults. Whether under attentional load (Anobile, Cicchini, & Burr, 2012), judging the randomness of numbers on a number line (Viarouge, Hubbard, Dehaene, & Sackur, 2010), or moving their hand to the number line to make an estimate (Dotan & Dehaene, 2013), adults also show a strong logarithmic component to their estimates. The dominant interpretation of these results is that unsupervised number-line estimates reflect the way that the brain encodes numerosity, with neural responses in the intraparietal sulcus (IPS) showing both a logarithmic-like tuning to a preferred numerosity and high activity during number-line estimation (Nieder, 2005; Nieder & Merten, 2007; Piazza, Izard, Pinel, Le Bihan, & Dehaene, 2004; Vogel, Grabner, Schneider, Siegler, & Ansari, 2013).

In this paper, we tested a challenge to the conventional interpretation of logarithmic-encoding of numerosity (Cicchini, Anobile, & Burr, 2014). According to this dynamic encoding hypothesis, the logarithmicity of numerosity estimates is not the default encoding pattern, but occurs as a response to the previous distribution of numbers encountered (specifically, overestimating the size of a number after encountering a large number and underestimating the size of a number encountering a small number). A critical prediction of this hypothesis is that estimates of numerosity should be unbiased prior to encountering any other estimates (such as on the first trial of a number line task) and that the central tendency of judgment should correlate with the logarithmic response pattern.

Against the dynamic encoding hypothesis, however, our data suggest that a dynamic encoding mechanism is neither necessary nor sufficient for logarithmic estimates of numerosity. If dynamic encoding were necessary for logarithmic estimates, the logarithmic component of estimates would be expected to be near zero prior to any previous numbers encountered. However, in both Study 1 and Study 2, we found that the logarithmic component on the first trial ranged from .6 to .7 (in adults) to 1.00 (in children). This result is consistent with the idea that the default perception of numerosity is Fechnerian, but not at all consistent with idea that compression requires a dynamic response.

Additionally, we found the central tendency of judgment effect was inconsistent and unrelated to logarithmicity of estimates. In Study 1, for example, a strong central tendency of judgment was evident after trial 1, yet it was not sufficient to elicit logarithmic responding. Rather, as subjects viewed more and more numbers, the logarithmicity of estimates *decreased* rather than *increased*. Further, although we replicated the central tendency of judgment effect in Study 1, where subjects encountered the same numerosity multiple times, we were unable to find the same tendency in Study 2, where subjects encountered each numerosity only once. This finding is not at all consistent with Ciccini et al.'s (2014) characterization of the central tendency of judgment as a domain general principle of perceptual judgments. Most importantly, however, the appearence (in Study 1) or non-appearance (in Study 2) of the central tendency of judgment effect was unrelated–or inversely related–to the logarithmicity of estimates, which was essentially zero in Study 1 and only .11 in Study 2. Overall, these results are wholly inconsistent with the idea that the logarithmicity of numerosity estimates is caused by a dynamic encoding mechanism that produces a central tendency of judgment effect.

Why might we observe a logarithmic-to-linear shift unfolding over trials if not due to a central tendency of judgment? At least two possibilities appeared likely. One possibility–consistent with Anobile et al. (2012)–is that subjects' attention to the task improves over time, thereby leading to increasingly accurate (linear) answers. Consistent with this possibility, we found that trial number strongly correlated with accuracy, such that the percent absolute error of adults' estimates decreased as the experiment progressed ($r(58) = -.45, p < .001$). On the other hand, we found no evidence of an increase in linearity of children's estimates, which had a logarithmic component of 1 on nearly every trial of the task. Nor does this account explain why a large logarithmic component (.60 to .70) was present among adults on the first trial.

In our view, a better explanation for our pattern of data was also suggested by Anobile et al. (2012), where they hypothesized that "both linear and compressed maps can coexist" and that "withdrawing attention may reveal a more native representation of number" (Anobile et al., 2012, p. 458). This view can certainly explain why adults' estimates start logarithmic and end linear. The coexistence of linear and compressed maps are also present in developmental data, where the same child provides either logarithmic or linear patterns of estimates depending on the familiarity of the numbers (e.g., estimating 50 to fall at 50 % of a 0-100 line, but estimating 50 to fall at substantially more than 5 % of a 0-1000 line, Siegler & Opfer, 2003). Finally, this view is remarkably consistent with the view originally endorsed by Siegler and Opfer (2003): "we believe that (a) individual children know and use multiple representations of numerical quantity over a period of many years; (b) with development, children rely on formally appropriate representations in an increasing range of numerical contexts; and (c) the numerical context influences childrens choice of representation" (pp. 237 - 238). Rather than challenging this account, we think the data on the dynamic encoding of numerosity is remarkably consistent with it.

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